Viability and Invariance of Sets under Transport Control Systems in Wasserstein Spaces

Helene Frankowska[1] and Benoît Bonnet-Weil[2]

[1]IMJ-PRG, CNRS and Sorbonne Université, Paris, France
[2] Laboratoire des Signaux et Systèmes, CNRS and CentraleSupélec, Gif-sur-Yvette, France

Abstract

The theory of meanfield control had a rapid recent development due to growing occurrences of control-theoretic questions in the context of large systems of interacting agents. Broadly speaking, the term meanfield control covers a large range of models in which a centralised entity aims at stirring a large number of agents so as to achieve a desired goal.

In this talk I will discuss some invariance properties of subsets of probability measures under controlled transport equations. More precisely, I will speak about viability and invariance of proper subsets of Wasserstein spaces $\mathcal{P}_p(R^d)$ of Borel probability measures with finite $p-$momentum and $p \geq 1$. The dynamical system is a nonlocal one, described by the transport control system

$$\partial_t \mu(t) + div(f(\mu(t), u(t))\mu(t)) = 0, \quad \mu(0) = \mu_0,\ u(t) \in U$$

where $f : \mathcal{P}_p(R^d) \times U \to Lip(R^d, R^d)$, $Lip(R^d, R^d)$ denotes the vector space of bounded Lipschitz maps from R^d into itself, U is a compact metric space, $\mu_0 \in \mathcal{P}_p(R^d)$ and controls are Lebesgue measurable functions $u : R_+ \to U$.

A subset $Q \subset \mathcal{P}_p(R^d)$ is called viable under the transport control system if for every $\mu_0 \in Q$, there exists a solution $\mu(\cdot)$ such that $\mu(t) \in Q$ for all $t \geq 0$. Q is called invariant under the transport control system if every such solution satisfies $\mu(t) \in Q$ for all times $t \geq 0$. To characterise these two properties in the spirit of the classical results an analogue of tangents to Q is introduced.

Two cases have to be distinguished. When $p > 1$ the characterisations are stronger and follow from some duality arguments [1]. In contrast for $p = 1$ the Euler approximation scheme is applied [2] to get viable solutions.

References

[1] Bonnet B. & Frankowska H.; *Viability and invariance of proper sets for continuity inclusions in Wasserstein spaces*, SIMA, 56 (2024), 2863-2914.

[2] Bonnet B., Corella A. & Frankowska H.; *Viability theorem in 1-Wasserstein space*, submitted, 2025.